

Contractions on the Classical Double

Angel Ballesteros¹ and Mariano A. del Olmo²

¹ *Departamento de Física,
Universidad de Burgos. E-09001, Burgos, Spain*

² *Departamento de Física Teórica,
Universidad de Valladolid. E-47011, Valladolid, Spain.*

Abstract

Lie algebra contractions on the classical Drinfel'd Double of a given Lie bialgebra are introduced and compared to the usual Lie bialgebra contraction theory. The connection between both approaches turns out to be intimately linked to duality problems. The non-relativistic (Galilean) limit of a (1+1) Poincaré Double is used to illustrate the contraction process. Finally, it is shown that, in a certain sense, the classical limit in a quantum algebra can be thought as a certain contraction on the corresponding Double.

1 Introduction

The well known result of Drinfel'd [1] that establishes an one to one correspondence between Poisson-Lie groups and Lie bialgebras is the key stone to understand the essential role that Lie bialgebras play in the quantization of Poisson-Lie structures (quantum groups). The concept of classical double is just a reformulation of that of Lie bialgebra in terms of a (double) dimensional Lie algebra. This “duplication” process can be iterated by taking into account that the double of a Lie bialgebra can be in turn equipped with a Lie bialgebra structure by means of a canonical r -matrix. The quantization of such a double Lie bialgebra is the so-called quantum double, a structure that has been essential in the explicit obtention of many quantum R -matrices. (See [2] for a detailed exposition and references therein).

On the other hand, the initial developements in quantum groups were mainly devoted to the construction of deformations of semisimple Lie structures [3], and it was soon discovered that contraction methods allowed to construct quantum deformations of non-semisimple algebras in a very efficient way [4]. Since then, contraction techniques have been extensively used to generate quantum deformations of Poincaré and Galilei groups and algebras (see, for instance, [5, 6, 7]).

More recently, a systematic approach to the contractions of quantum groups and algebras has been introduced [8, 9, 10]. That scheme is based in a contraction theory for the underlying Lie bialgebras, that turn out to be again the objects characterizing the behaviour of the full quantum structures.

The aim of this paper is to explore how contractions can be implemented onto the corresponding classical doubles and to show that the resultant approach generalizes the previous ones and allows the study of new processes as the classical limit in a contraction context. In Section 2 we shall briefly review the notion of the classical double and in Section 3 the notions of Lie algebra and Lie bialgebra contraction will be recalled. General contractions on the double Lie algebra D of an arbitrary Lie bialgebra (g, η) will be characterized in Section 4. In this respect it will be essential to consider how the internal pairing among generators of the double behaves under contraction. The solution to this problem will lead to us to recover the Lie bialgebra contraction theory. Throughout the paper, the approach will be illustrated by studying how the classical double of a Lie bialgebra of the two-dimensional Euclidean algebra can give rise to a (1+1) Galilei double by means of a suitable contraction that implements the non-relativistic limit.

2 The classical double of a Lie bialgebra

The notion of Lie bialgebra (g, η) arises when a Lie algebra g is furnished with a cocommutator $\eta : g \rightarrow g \otimes g$ such that η is a 1-cocycle and the dual map $\eta^* : g^* \otimes g^* \rightarrow g^*$ is a Lie bracket on g^* (the dual vector space of g). A Lie bialgebra (g, η) is called a coboundary bialgebra if there exists an element $\rho \in g \otimes g$ (the classical r -matrix), such that

$$\eta(X) = [1 \otimes X + X \otimes 1, \rho], \quad \forall X \in g. \quad (2.1)$$

On the other hand, the map (2.1) defined by using an arbitrary $\rho \in g \otimes g$ defines a Lie bialgebra if and only if the symmetric part of ρ is g -invariant and the antisymmetric part of ρ is a solution of the generalized Classical Yang-Baxter Equation (CYBE) [1].

Let (\mathcal{G}, η) be a Lie bialgebra, and $\{X_i\}$ a basis of g . With such a basis, the Lie bialgebra can be characterized by the pair of structure tensors (c_{ij}^k, f_n^{lm}) that define the commutator and the cocommutator in the form:

$$[X_i, X_j] = c_{ij}^k X_k, \quad \eta(X_n) = f_n^{lm} X_l \otimes X_m. \quad (2.2)$$

In this language, the cocycle condition becomes the following compatibility condition between the tensors c and f

$$f_k^{ab} c_{ij}^k = f_i^{ak} c_{kj}^b + f_i^{kb} c_{kj}^a + f_j^{ak} c_{ik}^b + f_j^{kb} c_{ik}^a. \quad (2.3)$$

Let us now consider a basis $\{x^i\}$ of g^* such that $\langle x^i, X_j \rangle = \delta_j^i$; then (\mathcal{G}^*, ξ) is also a Lie bialgebra with structure tensors (f, c) , i.e.,

$$\{x^i, x^j\} = f_k^{ij} x^k, \quad \xi(x^n) = c_{lm}^n x^l \otimes x^m. \quad (2.4)$$

This intrinsic self-dual character between the two Lie algebras included within a Lie bialgebra lead to the consideration of the pair $(\mathcal{G}, \mathcal{G}^*)$ and its associated vector space $\mathcal{G} \oplus \mathcal{G}^*$, that can be endowed with a Lie algebra structure by means of the commutators

$$[X_i, X_j] = c_{ij}^k X_k, \quad [x^i, x^j] \equiv \{x^i, x^j\} = f_k^{ij} x^k, \quad [x^i, X_j] = c_{jk}^i x^k - f_j^{ik} X_k. \quad (2.5)$$

This Lie algebra, $\mathcal{D}(\mathcal{G})$, is called the Double Lie algebra of (G, η) . Obviously, \mathcal{G} and \mathcal{G}^* are subalgebras of \mathcal{D} , and the compatibility condition (2.3) is just the Jacobi identity for (2.5).

Moreover, if \mathcal{G} is a finite dimensional Lie algebra, then \mathcal{D} is a (coboundary) Lie bialgebra with classical r -matrix

$$r = X_i \otimes x^i, \quad (2.6)$$

fulfilling the Classical Yang-Baxter Equation (CYBE). The cocommutator $\delta(r)$ derived from (2.6) is

$$\delta(X_i) = f_i^{jk} X_j \wedge X_k, \quad \delta(x^i) = c_{jk}^i x^j \wedge x^k. \quad (2.7)$$

To summarize, we can say that \mathcal{D} can be seen as both a Lie algebra and a (triangular) coboundary Lie bialgebra. In fact this “double Lie bialgebra” has as sub-Lie-bialgebras the original one (\mathcal{G}, η) and its dual. This duplication process can be obviously iterated. It is also worthy to recall the following inner product on $\mathcal{G} \oplus \mathcal{G}^*$ [2]:

$$\langle X_i, X_j \rangle = 0, \quad \langle x^i, x^j \rangle = 0, \quad \langle x^i, X_j \rangle = \delta_j^i, \quad \forall i, j. \quad (2.8)$$

This pairing is essential when Lie bialgebras are connected with Poisson-Lie groups. Then, the x^i generators are interpreted as local coordinates on the Lie group with Lie algebra G , and the tensor f_k^{ij} gives a Poisson bracket among this coordinates. Such a bracket is just the linear part of the full Poisson-Lie structure on the Lie group associated to Lie bialgebra (G, η) .

• *Example 1.* Let us now explicitly consider the Euclidean Lie algebra $e(2)$ with Lie brackets

$$[J_{12}, P_1] = P_2, \quad [J_{12}, P_2] = -P_1, \quad [P_1, P_2] = 0. \quad (2.9)$$

The cocommutator

$$\eta(J_{12}) = z J_{12} \wedge P_2, \quad \eta(P_1) = z P_1 \wedge P_2, \quad \eta(P_2) = 0, \quad (2.10)$$

endows $e(2)$ with a one-parameter family of Lie bialgebra structures. Let us write explicitly the Double Lie algebra linked to it by writing the generators of $e(2)^*$ as $\{j_{12}, p_1, p_2\}$:

$$\begin{aligned} [J_{12}, P_1] &= P_2, & [J_{12}, P_2] &= -P_1, & [P_1, P_2] &= 0, \\ [j_{12}, p_1] &= 0, & [j_{12}, p_2] &= z j_{12}, & [p_1, p_2] &= z p_1, \\ [j_{12}, J_{12}] &= -z P_2, & [j_{12}, P_1] &= 0, & [j_{12}, P_2] &= 0, \\ [p_1, J_{12}] &= -p_2, & [p_1, P_1] &= -z P_2, & [p_1, P_2] &= j_{12}, \\ [p_2, J_{12}] &= p_1 + J p_{12}, & [p_2, P_1] &= -j_{12} + z P_1, & [p_2, P_2] &= 0. \end{aligned} \quad (2.11)$$

Note that we have assumed a commutator notation for the Lie bracket in the Double. We have also explicitly preserved the parameter z within the original cocommutator. Such a parameter will be identified with the deformation parameter of the Euclidean quantum algebra that has (2.10) as the first order deformation of the quantum coproduct. In this context, the explicit consideration of z will appear as relevant when contractions at the quantum algebra level are discussed (see [8]).

As we have commented before, the six dimensional algebra (2.11) admits, by construction, a coboundary Lie bialgebra structure δ that can be deduced from the r-matrix (2.6) (in this case, $r = J_{12} \otimes j_{12} + P_1 \otimes p_1 + P_2 \otimes p_2$):

$$\begin{aligned} \delta(J_{12}) &= z J_{12} \wedge P_2, & \delta(P_1) &= z P_1 \wedge P_2, & \delta(P_2) &= 0, \\ \delta(j_{12}) &= 0, & \delta(p_1) &= p_2 \wedge j_{12}, & \delta(p_2) &= j_{12} \wedge p_1. \end{aligned} \quad (2.12)$$

3 Lie algebra and Lie bialgebra contractions

We present in this section a brief review about the basic ideas related with the contraction of Lie algebras and the extension of this procedure to the case of Lie bialgebra structures.

3.1 Lie algebra contractions

Let us start by remembering the concept of contraction of Lie algebras according with the Saletan approach [11].

Let (A, m) and (A', m') be two algebras with the same underlying vector space V and products m and m' , i.e., $m : V \otimes V \rightarrow V$. Let us assume that there exists a continuous uniparametric family ϕ_ε of linear mappings

$$\phi_\varepsilon : V \rightarrow V, \quad \varepsilon \in (0, 1], \quad \phi_\varepsilon|_{\varepsilon=1} = id, \quad (3.1)$$

such that ϕ_ε is invertible when $\varepsilon \neq 0$ and singular when $\varepsilon = 0$. The algebra (A', m') is said to be a contraction of (A, m) if m' can be defined as

$$m' = \lim_{\varepsilon \rightarrow 0} m_\varepsilon = \lim_{\varepsilon \rightarrow 0} \phi_\varepsilon^{-1} \circ m \circ (\phi_\varepsilon \otimes \phi_\varepsilon). \quad (3.2)$$

When A and A' are Lie algebras with Lie brackets m and m' , the expression (3.2) can be written as

$$[X, Y]' := \lim_{\varepsilon \rightarrow 0} \phi_\varepsilon^{-1} [\phi_\varepsilon(X), \phi_\varepsilon(Y)]. \quad (3.3)$$

In the case that $V = H \oplus W$, such that H is the underlying vector space of a subalgebra of A and W is the supplement of H , and the mapping ϕ_ε is defined by

$$\phi_\varepsilon|_H = id, \quad \phi_\varepsilon|_W = \varepsilon id. \quad (3.4)$$

we shall say that we have performed an Inönü–Wigner [12] contraction along the subalgebra H .

An interesting generalization of the IW contraction can be defined as follows. Let g be a Lie algebra whose associated vector space V is written as a direct sum of vector subspaces

$$V = \bigoplus_i V_i, \quad i = 0, 1, \dots, N \geq 1. \quad (3.5)$$

The mapping ϕ_ε will be called generalized IW contraction [13] if

$$\phi_\varepsilon|_{V_i} = \varepsilon^{n_i} Id|_{V_i}, \quad 0 \leq n_0 < n_1 < n_2 < \dots < n_N, \quad n_i \in \mathbb{R}. \quad (3.6)$$

It can be shown that a given Lie algebra admits a generalized IW contraction if and only if

$$[V_i, V_j] \subset \bigoplus_k V_k, \quad (3.7)$$

where by (3.7) we understand that a given subspace V_k can originate a contribution to the right hand side of the bracket if $n_k \leq n_i + n_j$.

Under a generalized IW contraction the structure constants of the contracted Lie algebra can be written in terms of the original ones as

$$c'_{ij}{}^k = \lim_{\varepsilon \rightarrow 0} \varepsilon^{n_i + n_j - n_k} c_{ij}^k, \quad (3.8)$$

since ϕ_ε acts on any generator X in V_i as $\phi_\varepsilon(X) = \varepsilon^{n_i} X$.

A further generalization of the above kind of contractions can be introduced by considering negative values of the n_i exponents [14]. Finally, we recall that the so-called graded contractions, that were introduced in the early nineties by Montigny, Moody and Patera [15], extended these concepts to include arbitrary modifications of the structure constants of a given algebra compatible with a fixed grading.

3.2 Lie bialgebra contractions

Let (g, η) be a Lie bialgebra and let g' be a Lie algebra obtained from g by means of a (generalized) IW contraction ϕ_ε . If n is a positive real number such that the limit

$$\eta' := \lim_{\varepsilon \rightarrow 0} \varepsilon^n (\phi_\varepsilon^{-1} \otimes \phi_\varepsilon^{-1}) \circ \eta \circ \phi_\varepsilon \quad (3.9)$$

exists, then is possible to prove that (g', η') is a Lie bialgebra [8]. In fact, there exists a unique minimal value f_0 of n such that, if $n \geq f_0$ the limit (3.9) exists, and if $n > f_0$ that limit is zero.

We will say that (g', η') is a contracted Lie bialgebra of (g, η) . The pair (ϕ_ε, n) will be called a Lie bialgebra contraction (or bicontraction). The minimal value f_0 that ensures the existence of (3.9) will be called the fundamental contraction constant of (\mathcal{G}, η) associated to the contraction ϕ_ε , and (ϕ_ε, f_0) will be called a fundamental bicontraction.

These results can be extended to the case of coboundary Lie bialgebras [8], i.e., the contraction of the classical r -matrix can be considered. If $(g, \eta(\rho))$ is a coboundary Lie bialgebra with ρ its classical r -matrix and g' is a Lie algebra obtained by means of a (generalized) IW contraction ϕ_ε from g , we can study the limit

$$\rho' := \lim_{\varepsilon \rightarrow 0} \varepsilon^n (\phi_\varepsilon^{-1} \otimes \phi_\varepsilon^{-1})(\rho). \quad (3.10)$$

One can prove that if there is a positive real number n such that the limit (3.10) exists, then $(g', \eta'(\rho'))$ is a coboundary Lie bialgebra. Also in this case, there is a unique minimal value c_0 of n such that, if $n \geq c_0$ (3.10) exists and, if $n > c_0$ such a limit is zero.

The minimal value c_0 that ensures the existence of the limit (3.10) will be called coboundary contraction constant of the Lie bialgebra $(g, \eta(\rho))$ relative to the contraction mapping ϕ_ε . The pair (ϕ_ε, c_0) is called a coboundary bicontraction of $(g, \eta(\rho))$ associated to ϕ_ε . Moreover, the relation $f_0 \leq c_0$ is always fulfilled for a given $(g, \eta(\rho))$ and ϕ_ε .

Complete proofs of all the statements included in this section can be found in [9] as well as various examples (see also [8, 10]).

4 Contracting the classical double

Since \mathcal{D} is a Lie algebra, the theory developed in subsection 2.1 can be fully applied onto \mathcal{D} in order to obtain new Lie algebras of the same dimension. However, it would be interesting to know how it is possible to characterize the contractions on \mathcal{D} that give rise to a double Lie algebra.

4.1 Double-preserving contractions of \mathcal{D}

We shall say that a contraction of \mathcal{D} is a “double-preserving” contraction, if the contracted algebra \mathcal{D}' is a classical double, i.e., if it preserves the internal structure given by (2.5).

Let us consider the most arbitrary generalized Doebner-Melsheimer contraction constructed by considering that each subspace is spanned by only one generator of \mathcal{D} . This means that the contraction mapping is defined as

$$\phi_\varepsilon(X_i) = \varepsilon^{m_i} X_i, \quad \phi_\varepsilon(x^i) = \varepsilon^{n_i} (x^i), \quad (4.1)$$

where $m_i, n_i \in \mathbb{Z}$. If we use now the definition (3.3) to contract the Double commutation rules (2.5), we shall obtain the following contracted structure tensors c' and f' :

$$\begin{aligned} c'_{ij}{}^k &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{m_i+m_j-m_k} c_{ij}^k, & f'_k{}^{ij} &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{n_i+n_j-n_k} f_k^{ij} \\ c'_{ij}{}^k &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{m_i-n_j+n_k} c_{ij}^k, & f'_k{}^{ij} &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{n_i-m_j+m_k} f_k^{ij}. \end{aligned} \quad (4.2)$$

The expressions in the first row come from the contraction of commutators $[X_i, X_j]$ and $[x^i, x^j]$, respectively; and those of the second row from $[x^i, X_j]$. If we impose the c' and f' components to be the same whenever they appear in the contracted Double, we are lead to the conditions

$$m_i - m_j = -(n_i - n_j), \quad (4.3)$$

for all couples (i, j) such that there exists either a non vanishing c_{ij}^k or f_k^{ij} component in the original Double (when a given component of any of the original tensors vanishes, there is no consistency constrain in the contracted limit of such a component, that will be zero in any case -see (4.2)-).

Afterwards, in order to avoid divergences in the limit $\varepsilon \rightarrow 0$ of (4.2) we shall have to impose that, for all triads (i, j, k) labelling a non-vanishing component either of c or of the tensor f

$$m_i + m_j - m_k \geq 0, \quad n_i + n_j - n_k \geq 0. \quad (4.4)$$

This is tantamount to say that we need the generalized IW contraction (4.1) to be a good contraction for both g and g^* Lie algebras. Conditions (4.3) and (4.4) will therefore suffice to guarantee that \mathcal{D}' is a classical double.

- *Example 2.* A quite simple Double-preserving contraction is obtained if we define $m_i = n_i = N \in \mathbb{Z}^+$, $\forall i$. This contraction always fulfills (4.3) and (4.4) and originates an abelian contracted algebra since all the structure constants vanish after the limit $\varepsilon \rightarrow 0$.

- *Example 3.* Let us now consider the two dimensional Euclidean double introduced in Example 1 and let us try to implement on it the non-relativistic limit. It is well known that, at the Lie algebra level, such a limit is equivalent to the following IW contraction

$$\phi_\varepsilon(J_{12}) = \varepsilon J_{12}, \quad \phi_\varepsilon(P_1) = P_1, \quad \phi_\varepsilon(P_2) = \varepsilon P_2, \quad (4.5)$$

where $\varepsilon = 1/c$, with c the speed of light. We have to find now how the contraction mapping ϕ has to act on the remaining x^i generators of \mathcal{D} in order to obtain a correct non-relativistic limit on the double.

If we consider that $\langle J_{12} \rangle = V_1, \langle P_1 \rangle = V_2$ and $\langle P_2 \rangle = V_3$, we shall have that, from (4.1), $m_1 = m_3 = 1$ and $m_2 = 0$. This notation immediately implies the definition of the remaining subspaces as $\langle j_{12} \rangle = v_1, \langle p_1 \rangle = v_2$ and $\langle p_2 \rangle = v_3$. Now we can say that the structure tensors on the Euclidean double only have as the only non-vanishing components $c_{12}^3, c_{13}^2, f_1^{13}, f_2^{23}$ and their skew-symmetric counterparts. This fact leads us to impose onto the n_i exponents the conditions (4.3):

$$\begin{aligned} m_1 - m_2 &= n_2 - n_1 = 1, \\ m_1 - m_3 &= n_3 - n_1 = 0, \\ m_2 - m_3 &= n_3 - n_2 = -1. \end{aligned} \tag{4.6}$$

The obvious set of solutions for these equations is

$$n_1 = \alpha, \quad n_2 = 1 + \alpha, \quad n_3 = \alpha, \quad \alpha \in \mathbb{Z}. \tag{4.7}$$

Now we have to go to the remaining conditions (4.4), that are translated into the following inequalities, each of them coming from a triad (i, j, k) labelling a not-vanishing tensor component:

$$\begin{aligned} m_1 + m_2 - m_3 &\geq 0, & m_1 + m_3 - m_2 &\geq 0, \\ n_1 + n_3 - n_1 &\geq 0, & n_2 + n_3 - n_2 &\geq 0. \end{aligned} \tag{4.8}$$

The two first ones are obviously fulfilled by the initial values of m_i (we started from a right contraction of the Euclidean Lie algebra). The second ones are indeed equivalent to write $n_3 = \alpha \geq 0$. Therefore only generalized IW contractions are allowed in this case. Moreover, it can be easily checked that if $\alpha > 0$, all the components of f' vanish, thus obtaining a Double algebra of a (1+1) Galilei bialgebra with trivial cocommutator. On the other hand, if we compute the full contracted algebra when $\alpha = 0$, i.e., with contraction mapping given by (4.5) and

$$\phi_\varepsilon(j_{12}) = j_{12}, \quad \phi_\varepsilon(p_1) = \varepsilon p_1, \quad \phi_\varepsilon(p_2) = p_2, \tag{4.9}$$

we obtain a expression for a classical Galilei double that differs with respect to (2.11) in the following commutation rules:

$$[J_{12}, P_2] = 0, \quad [p_1, J_{12}] = 0, \quad [p_1, P_2] = 0. \tag{4.10}$$

In fact, we have just obtained the double of a Galilei Lie bialgebra that preserves under contraction the same cocommutator (2.10). As we have proven by the previous analysis, this is the double corresponding to the only non-trivial Galilei Lie bialgebra that can be reached from our Euclidean one by using the non-relativistic limit.

4.2 The duality problem

An important property of the double is the pairing $\langle x^i, X_j \rangle = \delta_j^i$. In what follows we shall analyse the implications of a duality preservation during the contraction process.

Let ϕ_ε be the contraction mapping defined by (4.1). Then we shall say that the pairing $\langle x^i, X_j \rangle$ will be preserved under contraction if the “contracted generators” $\phi_\varepsilon(x^i), \phi_\varepsilon(X_j)$ fulfill

$$\langle \phi_\varepsilon(x^i), \phi_\varepsilon(X_j) \rangle = \varepsilon^{n_i+m_j} \langle x^i, X_j \rangle = \delta_j^i, \quad (4.11)$$

therefore, we get the condition

$$n_i + m_i = 0. \quad (4.12)$$

And this means that we are forced to use negative powers in order to define contractions preserving the pairing. We thus deal now with Doebner-Melsheimer contractions [14].

Let us now suppose that we start from a known contraction of the original Lie algebra } that is a generalized IW one. This implies that all $m_i > 0$ and, therefore, all $n_i < 0$. This fact seems to indicate that we can find serious problems in the convergency of the $\varepsilon \rightarrow 0$ limit (or, equivalently, in solving conditions (4.4)).

These difficulties can be confirmed by making use of the Euclidean example we have worked out previously and considering the same contraction mapping (4.5) on the generators of the $e(2)$ algebra. If we want to preserve the pairing (2.8) under contraction, we are forced to have $n_1 = -1$, $n_2 = 0$ and $n_3 = -1$. Thus, α has to be -1 , a value which is forbidden by convergency conditions (4.8). In conclusion, for this classical double, and provided the contraction mapping for the Euclidean Lie algebra generators is given by (4.5), there does not exist any contraction (in the sense of the previous section) that preserves the pairing (in the sense of (4.11)).

A possibility in order to avoid such divergencies in the f' components is to use a “renormalization” factor ε^{t_0} , $t_0 \geq 0 \in \mathbb{R}$, such that $\forall N \geq t_0$, and for all the triads (i, j, k) coming from non-vanishing components of f , the new definition of the contracted components be

$$f_k'^{ij} := \lim_{\varepsilon \rightarrow 0} \varepsilon^{n_i+n_j-n_k+N} f_k^{ij}. \quad (4.13)$$

By construction, this contracted tensor does not diverge (of course, the exponent t_0 depends on both the mapping ϕ_ε and the double considered).

However, this way for contracting the double and preserving duality has to be shown to be consistent (i.e., we should prove now that definition (4.13) and the usual contraction of the tensor c give rise to a Lie algebra with classical double structure). Fortunately, further computations are not necessary because (4.13) is nothing but the rephrasing in the classical double language of the Lie bialgebra contraction theory of Section 2. It can be easily checked that t_0 is just the fundamental contraction constant f_0 , and that the right “renormalized” contraction gives always rise to the classical double corresponding to the contracted Lie bialgebra obtained by making use of the theory sketched in Section 3.2.

In particular, if this duality-preserving approach is applied onto the Euclidean double (2.11), we shall obtain that $t_0 = 1$. On the other hand, the Euclidean Lie bialgebra (2.9-2.10) can be contracted by means of a fundamental bicontraction characterized, as

expected, by $f_0 = 1$. Moreover, the Galilean double so obtained coincides with the one derived in the previous section by using a generalized IW contraction.

4.3 The classical limit as a contraction

Given a quantum algebra, the limit $z \rightarrow 0$ of the deformation parameter can be interpreted as the classical limit: under it, the quantum coproduct in the deformed algebra becomes cocommutative and, consequently, the algebra of functions on the group becomes commutative. At the classical double level, this procedure is equivalent to make zero all the components of the tensor f [16]. These assertions can be easily illustrated by using the Example 2.

It is remarkable that such a limit process can be described in a completely equivalent way as a pure IW contraction on the classical double along the Lie (sub)algebra g . Explicitly, let us consider an arbitrary classical double and the contraction mapping on it defined by

$$\phi_\varepsilon(X_i) = X_i, \quad \phi_\varepsilon(x^i) = \varepsilon x^i. \quad (4.14)$$

Obviously, since $m_i = 0$ and $n_i = 1$ for all i , this is a double-preserving contraction fulfilling conditions (4.3) and (4.4). It is also immediate to check that the contracted double has commutation rules

$$[X_i, X_j] = c_{ij}^k X_k, \quad [x^i, x^j] = 0, \quad [x^i, X_j] = c_{jk}^i x^k. \quad (4.15)$$

Note that another double-preserving contraction can be obtained by defining

$$\phi_\varepsilon(X_i) = \varepsilon X_i, \quad \phi_\varepsilon(x^i) = x^i. \quad (4.16)$$

This process annihilates the c tensor, and can be interpreted as the classical limit on the dual, that gives rise to a classical double corresponding to a Lie bialgebra structure on a commutative algebra.

Acknowledgments

This work has been partially supported by projects DGICYT/PB94-1115 and DGES/PB95-0719 from the Ministerio de Educación y Ciencia de España.

References

- [1] V.G.Drinfel'd, Sov. Math. Dokl. **27** (1983), 68.
- [2] V. Chari, A. Pressley, *A Guide to Quantum Groups*. Cambridge Univ. Press, Cambridge 1994.
- [3] N. Yu. Reshetikhin, L.A. Takhtadzhyan and L.D. Fadeev, Leningrad Math. J. **1** (1990), 193.

- [4] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, *Contractions of quantum groups*, Lecture Notes in Mathematics n. 1510. Springer-Verlag, Berlin (1992).
- [5] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, J. Math. Phys. **32** (1991), 1159.
- [6] J. Lukierski, H. Ruegg and A. Nowicky, Phys. Lett. **B293** (1992), 344.
- [7] A. Ballesteros, F.J. Herranz, M.A. del Olmo and M. Santander, J. Math. Phys. **35** (1994), 4928.
- [8] A. Ballesteros, N.A. Gromov, F.J. Herranz, M.A. del Olmo and M. Santander, J. Math. Phys. **36** (1995), 5916.
- [9] A. Ballesteros, *Contractions of Lie bialgebras and quantum deformations of kinematical symmetries*, Ph. D. Thesis (in Spanish), Universidad de Valladolid (1995).
- [10] A. Ballesteros and M.A. del Olmo, *Contractions of Poisson-Lie Groups, Lie Bialgebras and Quantum Deformations*, in “Quantum Groups and Quantum Spaces”, Warsaw (1996).
- [11] E. Saletan, J. Math. Phys. **2** (1961), 1.
- [12] E. İnönü and E.P. Wigner, Proc. Natl. Acad. Sci. U. S. **39** (1953), 510.
- [13] E. Weimar-Woods, J. Math. Phys **32** (1991), 2028.
- [14] H.D. Doebner, O.Melsheimer, Il Nuovo Cim. **A49** (1967), 306.
- [15] M. de Montigny, J. Patera, J. Phys. A: Math. Gen. **24** (1991), 525; R.J. Moody ,J. Patera, J. Phys. A: Math. Gen. **24** (1991), 2227.
- [16] F. Bonechi, R. Giachetti, E. Sorace and M. Tarlini, *Quantum double and differential calculi*, Lett. Math. Phys. **37** (1996), 405.